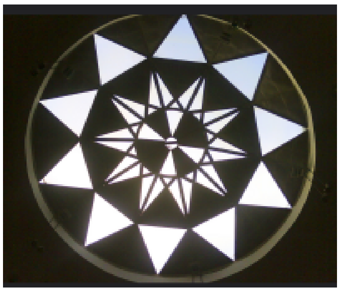
7.5 From Polygons to Circles

A Solidify Understanding Task

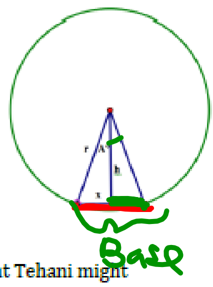


Part 1: From perimeter to circumference

In the previous task, *Planning the Gazebo*, you developed a strategy for finding the perimeter of a regular polygon with n sides inscribed in a circle of radius r. Tehani's strategy consists of the following formula:

$$P = 2n \cdot \underbrace{r\sin\left(\frac{360^\circ}{2n}\right)}_{\text{Base}}$$

Tehani drew this diagram as part of her work as she developed this formula.



1. Using Tehani's diagram, explain in detail how she arrived at her formula.

2. Since n is the only thing that varies in this formula, Travis suggests that Tehani might rewrite her formula in the form $P = 2r\left[n \cdot \sin\left(\frac{360^\circ}{2n}\right)\right]$. Because the perimeter of an n-gon approximates the circumference of a circle when n is a large number of sides, Travis suggests they examine what happens to the $n \cdot \sin\left(\frac{360^\circ}{2n}\right)$ portion of Tehani's formula as n gets larger and larger. Use a calculator or spreadsheet to complete the following table to see what happens.

n	$n \cdot \sin\left(\frac{360^\circ}{2n}\right)$
6	
12	
24	
48	
96	
100	
1,000	
10,000	

3. Write a formula for the circumference of a circle based on Tehani's formula for the perimeter of an inscribed regular n-gon and what you have observed while generating this table.

Part 2: From the area of a polygon to the area of a circle

Approach #1

Tehani's formula for the area of a regular polygon with n sides inscribed in a circle of radius r is:

R

$$A = n \cdot \underbrace{r\sin\left(\tfrac{360^\circ}{2n}\right)}_{\text{base}} \cdot \underbrace{r\cos\left(\tfrac{360^\circ}{2n}\right)}_{\text{height}}$$

4. Explain in detail how Tehani arrived at this formula. You may refer to the diagram above.

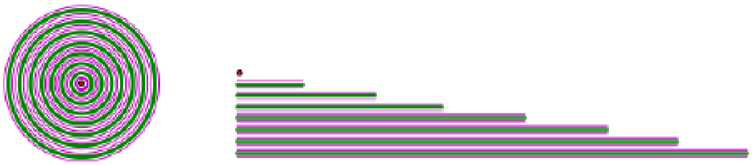
5. Travis suggests that they might rewrite Tehani's formula in the form $A = r^2 \cdot \left[n \cdot \sin\left(\tfrac{360^\circ}{2n}\right) \cdot \cos\left(\tfrac{360^\circ}{2n}\right)\right]$ and then examine what happens to the last part of the formula as n gets larger and larger. Use a calculator or spreadsheet to complete the following table and see what happens.

n	$n \cdot \sin\left(\tfrac{360^\circ}{2n}\right) \cdot \cos\left(\tfrac{360^\circ}{2n}\right)$
6	
12	
24	
48	
96	
100	
1,000	
10,000	

6. Write a formula for the area of a circle based on Tehani's formula for the area of an inscribed regular n-gon and what you have observed while generating this table.

Approach #2

A circle can be decomposed into a set of thin, concentric rings, as shown on the left in the following diagram. If we unroll and stack these rings we can approximate a triangle as shown in the figure on the right.



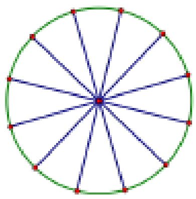
7. How might we describe the height of this "triangle" relative to the circle?

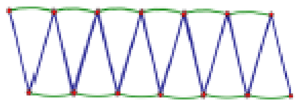
8. How might we describe the length of the base of this "triangle" relative to the circle?

9. As the rings get narrower and narrower the triangular shape gets closer and closer to an exact triangle with the same area as the circle. What would this diagram suggest for the formula of the area of a circle?

Approach #3

A circle can be decomposed into a set of congruent sectors, as shown on the left in the following diagram. We can rearrange these sectors to approximate a parallelogram as shown in the figure on the right.





10. How might we describe the height of the "parallelogram" relative to the circle?

11. How might we describe the base of this "parallelogram" relative to the circle?

12. As we decompose the circle into more and more sectors the "parallelogram" shape gets closer and closer to an exact parallelogram with the same area as the circle. What would this diagram suggest for the formula for the area of a circle?

SECONDARY MATH II // MODULE 7
CIRCLES A GEOMETRIC PERSPECTIVE - 7.5

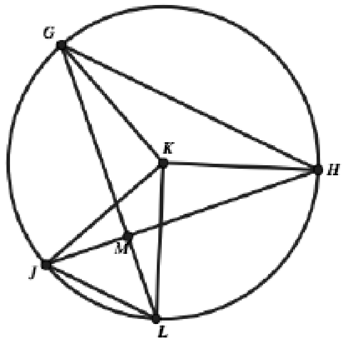
7.5

READY, SET, GO! Name Period Date

READY

Topic: Angles and Arcs of circles, ratios with similar shapes

Find the indicated values given the diagram and measurements provided below.

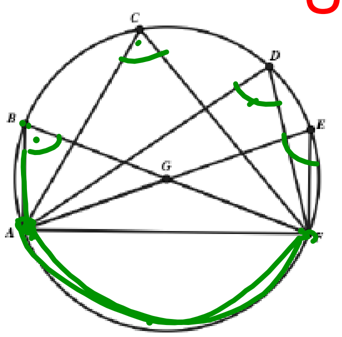


1. Given that $m\angle LGH$ and $m\angle GHJ$ are both 45°
What other measurement of angles or arcs do you know?
List them below (try to find six)

2. Given that ΔGKH has two sides that are radii of the circle.
What type of triangle is ΔGKH?
Are there any other triangles of this type in the diagram? If so, name them.

3. Given that the $m\widehat{GH}$ is 113.2°
What is $m\widehat{JL}$? (Look back at problems 1 and 2)

Use applet on website to help.



4. Given ⊙G, which angles would have the same measure? List them all below and say how you know they are equal.

$\angle ABF \cong \angle ACF \cong$

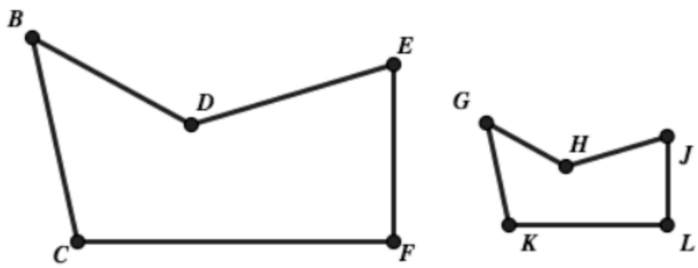
5. There are several triangles in the circle. List the triangles that are inscribed triangles.

Also, list any other triangles and classify as many of the triangles as you can.

6. Given that $m\angle GFE = 70°$ find all possible angle and arc measurements that you can.

Given the similar shapes below provide the desired missing sides or proportions.

7.



a. $\frac{BD}{BC} = \frac{?}{?}$

Fill in the proportion and state how you know it is correct.

b. $\frac{JL}{EF} = \frac{KL}{?}$

Fill in the proportion and state how you know it is correct.

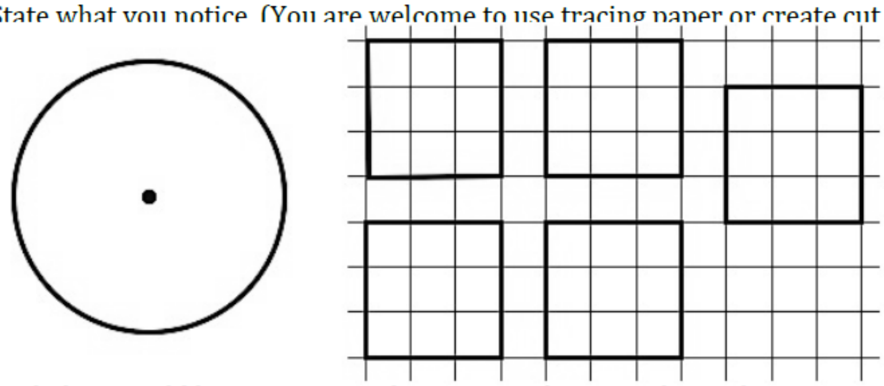
c. If possible, fill in the missing proportions so they are true statements. If not possible say why not.

i) $\frac{DE}{HJ} = \frac{?}{?}$ ii) $\frac{CF}{KL} = \frac{?}{?}$ iii) $\frac{GH}{?} = \frac{BD}{?}$ iv) $\frac{BC}{?} = \frac{GK}{?}$ v) $\frac{HJ}{?} = \frac{?}{BC}$

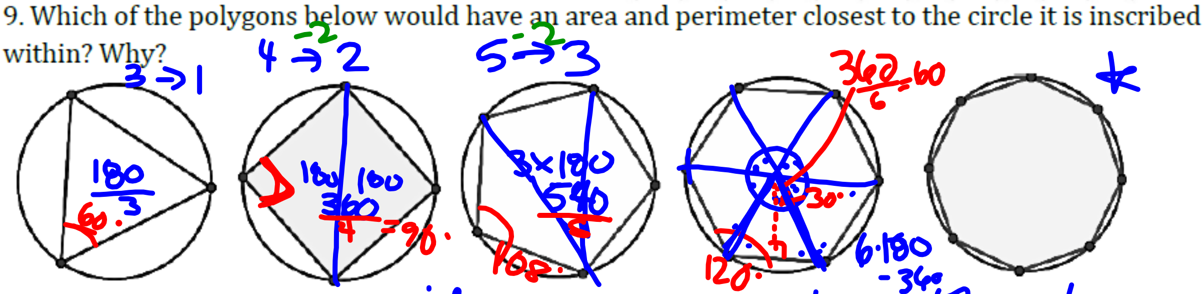
SET

Topic: Connecting polygons with circles

8. Below you are given a circle and also several squares that are constructed so that their sides are equal to the radius of the circle. Use these squares and circle to estimate how many squares it takes to fill in the area of the circle. State what you notice. (You are welcome to use tracing paper or create cut outs.)

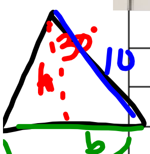


9. Which of the polygons below would have an area and perimeter closest to the circle it is inscribed within? Why?



The more sides a polygon has the closer and closer it is to being a circle

10. Given that the radius of the circles in **previous problem** is 10 feet. Find the area of each of the regular polygons and list them in the table below along with the measure of one angle for each polygon and the side length of each polygon. (A couple are filled in for you.)



Shape	One interior angle	Length of one side	Area of figure
Triangle	60°	17.32	129.9
Square	90°	$10\sqrt{2} = 14.14$	200
Pentagon	108		
Hexagon	120	10	261
Octagon			
Circle			

B $10\sin 30 = \frac{b}{10}$ $B=10$ H: $\cos 30 = \frac{h}{10}$ $h=8.7$ $A = \left(\frac{10 \cdot 8.7}{2}\right)6 = 261$

11. Show and explain how a circle can be cut into sectors and reconfigured to appear approximately as a polygon that could have its area calculated using a standard formula.

$$A = bh = \left(\frac{1}{2} \cdot 2\pi r\right) r = \pi r^2$$

12. Show and explain how a circle can be broken into several rings or interior circles that can be rearranged to appear approximately as a polygon that could have its area calculated using a standard formula.

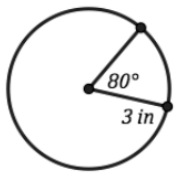
Answer: $A = \frac{bh}{2} = \frac{2\pi r \cdot r}{2} = \pi r^2$

GO

Topic: Finding arc length as a distance

Just as a circle can be broken into 360 sectors as a means for finding the area of any size sector. Similarly the circumference of a circle can be broken into 360 equivalent pieces as a means for finding the distance actually traveled along any arc of the circle.

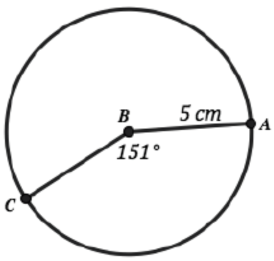
The circumference of the circle is 6π inches.
So, a sector of one degree would have length of $\frac{6\pi}{360}$ inches.
And so the area of the sector with a central angle of 80°
Would be $(80)\frac{6\pi}{360}$ which simplifies to be $\frac{4\pi}{3}$ inches or approximately 4.19 inches.



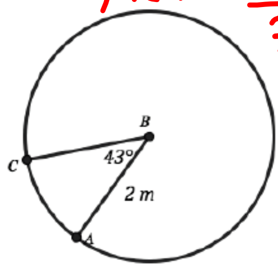
Look closely at the example on the previous page and then use this strategy for finding the arc length (actual distance traveled along the path of the arc) in each of the problems provided below.

Sector Area $= \frac{\theta}{360} \cdot (\pi r^2)$ Area

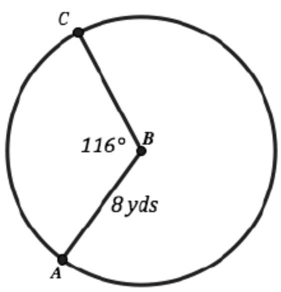
13.



14.



15.



16.