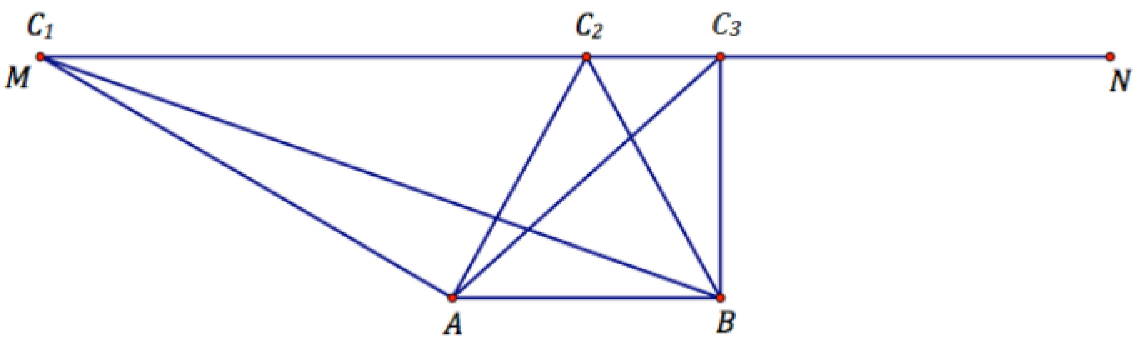
7.12 Cavalieri to the Rescue

A Solidify Understanding Task

Carlos, Clarita, and Zac are playing a geometry game. Each player selects a point C on the line segment $\overline{MN}$, which is parallel to line segment $\overline{AB}$. The points A, B and C form the vertices of a triangle. The player who creates the triangle with the largest area wins the game.

Carlos has placed his point at position C_1, Clarita has selected point C_2, and Zac has chosen to locate his point at C_3. Now they are discussing their choices before calculating the areas of each triangle to determine the winner.



Carlos: I chose my point so the triangle would stretch as far left as possible, enclosing a large amount of area.

Clarita: I thought it would be best to create an isosceles triangle so the triangle would be symmetric about its altitude, so I chose a point on segment MN directly above the midpoint of segment AB.

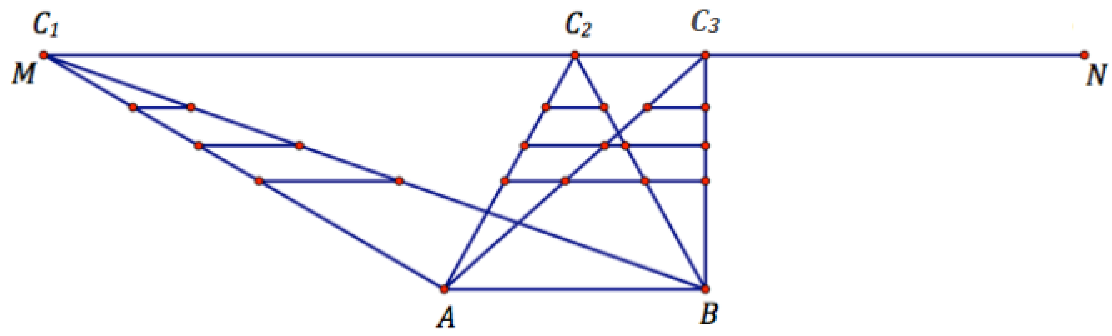
Zac: I thought that a right triangle would create the largest triangle, since my triangle would be half of a rectangle.

1. Without doing any calculations, who do you think created the triangle with the largest area?

2. If you were playing the game with Carlos, Clarita and Zac, where would you place your point C? Mark a point on segment MN to represent your best guess. You may mark your point at the same position as one that has already been chosen, if you agree that point would form the largest triangle.

3. Now it is time to determine a winner. Make any measurements necessary to calculate the winner of the game. Whose strategy won?

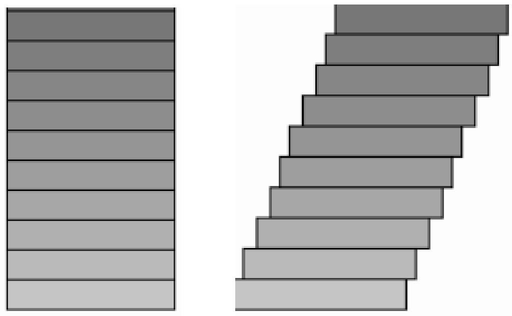
Carlos, Clarita and Zac were initially surprised by the results and wondered why the triangle images were so deceptive. They began to wonder if they could really believe their calculations. Then Carlos suggested an experiment. He drew a series of line segments in each triangle, with each segment parallel to the base of the triangle, $\overline{AB}$, and with corresponding segments in each of the triangles drawn at the same distance above the base, as shown in the diagram below. Carlos then measured each of the corresponding line segments.



4. Complete Carlos' experiment by measuring each of the corresponding line segments. What do you notice? What does this observation suggest about the areas of the triangles, and why?

Clarita said, "It feels like you are treating each triangle as if it was made up of a bunch of layers or slices."

Zac, inspired by Clarita's comment, pulled a handful of pennies out of his pocket and stacked them to form a cylinder. "I can calculate the volume of this stack of coins using the formula $V = Bh$. But what if I tilt the stack so it looks more like the Leaning Tower of Pisa. Now how do I figure out how much space the coins occupy?"

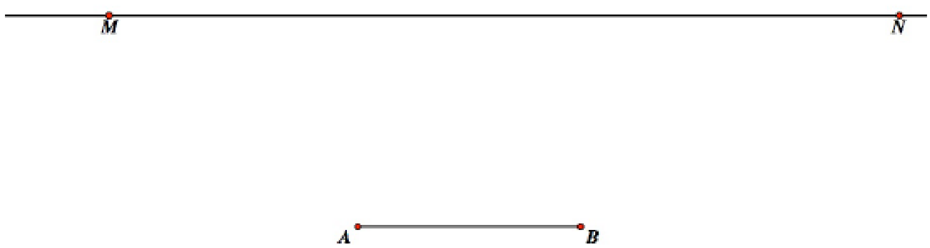


Carlos and Clarita smiled at Zac's clever way of illustrating an idea that was new to both of them. They were excited to tell their geometry teacher about their discovery and Zac's principle. They were surprised to hear that Zac wasn't the first person to think of it, and that it was known as Cavalieri's principle.

5. In your own words, state what you think Cavalieri's principle is, based on the triangle experiment and the stack of coins illustration.

Try out another experiment with Cavalieri's principle. Once again, line *MN* is parallel to segment *AB*. Measure the length of segment *AB*, and then mark a congruent segment *CD* anywhere on line *MN*. Connect the endpoints of segment *AB* to the endpoints of segment *CD* to form a parallelogram. Mark another segment *EF* on line *MN* so that *EF* is also congruent to *AB*. Connect the endpoints of segment *AB* to the endpoints of segment *EF* to form another parallelogram.

6. Use these two non-congruent parallelograms to illustrate Cavalieri's principle. What can you say about the areas of these two parallelograms, and what convinces you this is true?



While Zac's demonstration with the pennies has convinced Carlos and Clarita that the volume of prisms and cylinders where the parallel slices are not directly above each other is the same as corresponding ***right prisms*** and ***right circular cylinders***, they are wondering about pyramids and cones where the vertex of the cone is not directly above the center of the base. "How do you find the volume of these types of pyramids and cones?" Looking online, they have learned that these types of solids are called ***oblique prisms, oblique pyramids, oblique cylinders*** and ***oblique cones.***

While online, Carlos found this information: *The volume of a right prism or right circular cylinder is given by V = Bh, where B is the area of the surface that forms the base, and h is the height of the prism or cylinder. The volume of a pyramid or cone is 1/3 of the volume of a prism or cylinder with the same base and height.*

Clarita found this information: *The volume of a prism or cylinder is given by V = Bh, where B is the area of the congruent cross sections parallel to the base, and h is the height of the prism. The volume of a pyramid or cone is 1/3 of the volume of a prism or cylinder with the same base and height.*

7. How do these two statements differ and what do those differences imply?

Carlos, Clarita and Zac each find interesting geogebra animations and activities that give them additional insights about Cavalieri's principle and the volumes of oblique pyramids and cones.

Carlos finds a geogebra app that helps him visualize why the volume of a pyramid is $^1/_3$ of the volume of a prism with the same base (that is, the bases are congruent shapes) and height: https://www.geogebra.org/m/VNjkbxgE

Clarita finds a geogebra app that uses Cavalieri's principle to help her visualize why all pyramids with the same base (that is, the bases of each pyramid are congruent shapes) and height will have the same volume: https://www.geogebra.org/m/NwcBfSwZ#material/JNdR7RY7

Zac finds this geogebra app that uses Cavalieri's principle to derive the formula for the volume of a sphere. He is surprised to learn that Cavalieri's principle does not require that the parallel slices in the two shapes need to be congruent, only that they need to have the same area: https://www.geogebra.org/m/a9jQQQFz

8. Examine each of the geogebra apps that Carlos, Clarita and Zac found online, and summarize what you learn from each application. Be specific about how each app can suggest a mathematical argument for the claim, not just a visual one.

   a. Carlos' app:

   b. Clarita's app:

   c. Zac's app:

7.12

READY, SET, GO! Name Period Date

READY

Topic: Multiplying binomials and factoring quadratics into two linear factors.

Multiply the two factors and simplify by adding like terms.

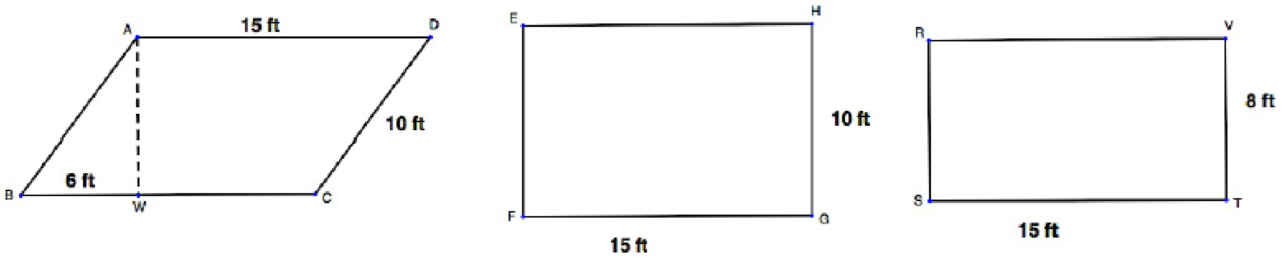
1. $(5x-7)(-3x+8)$ $-15x^2$	2. $(2x+3)(6x+1)$	3. $(-2x-7)(-7x-1)$
Factor each quadratic expression into two binomials.		
4. $10x^2+38x+36$ $2(5x^2+19x+18)$ 90; 9, 10; 19 $2(5x+9)(1x+2)$	5. $6x^2+17x-14$	6. $8x^2-2x-15$

SET

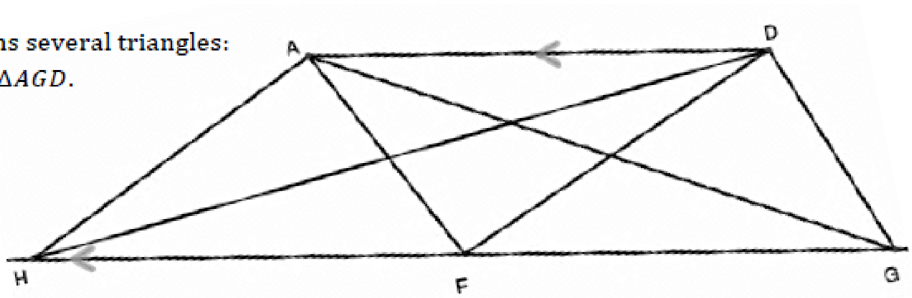
Topic: Applying Cavalieri's Theorem.

7. Below are 3 quadrilaterals. Compare the perimeters and the areas. How are they the same and how are they different?

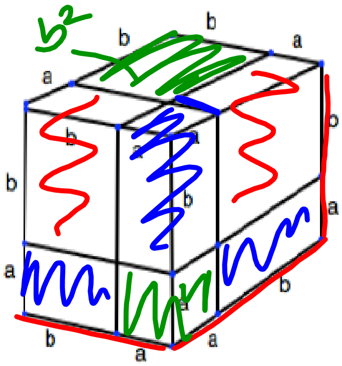
For which of the quadrilaterals are the calculations the same? Explain why this is happening.



8. The figure at the right contains several triangles: $\Delta HAF, \Delta HDF, \Delta HDG, \Delta AFD, and \Delta AGD$.
Given that $\overline{AD} \parallel \overline{HG}$, which triangles have the same area? Justify your answer.



9. The figure at the right shows a cube with edges of length $(a + b)$. The cube has been sliced into pieces by three planes parallel to its faces.



a. Write an expression for the volume of the cube in terms of a and b.

$V = (a+b)^3$

b. Into how many pieces is the cube cut?

8

c. How many of these pieces are also cubes?

2

d. Write an expression in terms of a and b for the volume of each cube you found.

b^3 and a^3

e. How many pieces have a volume of a^2b?

3

f. How many pieces have a volume of ab^2?

3

g. Write the volume of the figure as the sum of the volumes of its pieces.

$b^3 + a^3 + 3(a^2b) + 3(ab^2)$

h. Show the work that proves the equivalence between the factored and expanded forms.

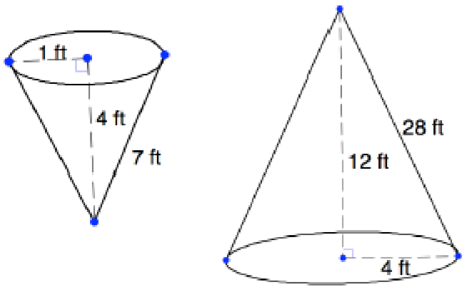
$(a+b)^3 = (a+b)(a+b)(a+b)$

GO

Topic: Congruent and Similar Solids

Determine if each pair of solids is similar, congruent, or neither. Justify your answers.

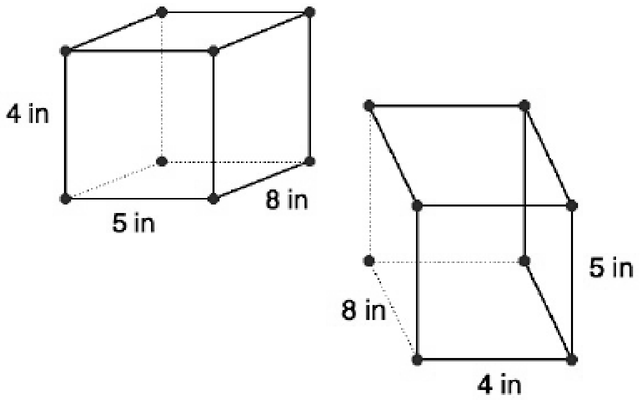
10.



Similar, congruent, or neither?

How do you know?

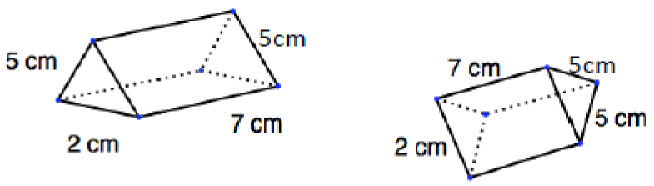
11.



Similar, congruent, or neither?

How do you know?

12.



Similar, congruent, or neither?

How do you know?