SECONDARY MATH II // MODULE 3
SOLVING QUADRATIC & OTHER EQUATIONS- 3.9

3.9 My Irrational and Imaginary Friends

A Solidify Understanding Task

Part 1: Irrational numbers

1. Verify that $4\left(x-\frac{5}{2}\right)\left(x+\frac{3}{2}\right)=0$ and $4x^2-4x-15=0$ are equivalent equations (show your work), and plot the solutions to the quadratic equations on the following number line:

$x=\frac{5}{2}\sim 2.5$ $\quad -\frac{3}{2}\sim -1.5$

-5 -4 -3 -2 -1 0 1 2 3 4 5

2. Verify that $(x-2+\sqrt{2})(x-2-\sqrt{2})=0$ and $4x^2-4x+2=0$ are equivalent equations (show your work), and plot the solutions to the quadratic equations on the following number line:

	x	-2	$+\sqrt{2}$
x	x^2	$-2x$	$\sqrt{2}x$
-2	$-2x$	4	$-2\sqrt{2}$
$-\sqrt{2}$	$-\sqrt{2}x$	$2\sqrt{2}$	-2

$\boxed{x^2-4x+2}$

$\sim 2+\sqrt{2}$
$\sim 2+1.41$
$\sim 3.41..$

$2-\sqrt{2}$
$2-1.411$
$.59$

$\sqrt{2}\cdot\sqrt{2}$
$-(\sqrt{2})^2$
-2

$2-\sqrt{2}$ $\quad 2+\sqrt{2}$

-5 -4 -3 -2 -1 0 1 2 3 4 5

You may have found it difficult to locate the exact points on the number line that represent the two solutions to the 2nd pair of quadratic equations given above. The following diagrams might help.

3. Find the perimeter of this isosceles triangle. Express your answer as simply as possible. perimeter

$1 + 1 + \sqrt{2}$

$\boxed{2 + \sqrt{2}}$

3.41

$2\sqrt{2} \neq$

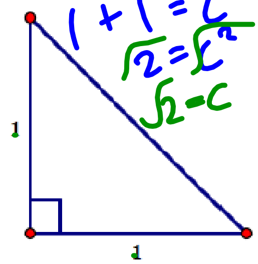
~ 2.8

$a^2 + b^2 = c^2$

$1 + 1 = c^2$

$\sqrt{2} = \sqrt{c^2}$

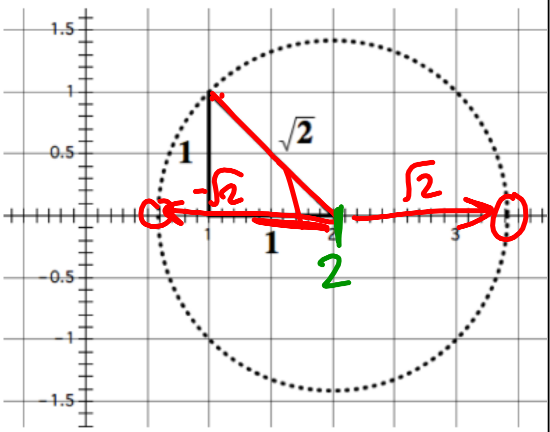
$\sqrt{2} = c$



We might approximate the perimeter of this triangle with a decimal number, but the exact perimeter is $2 + \sqrt{2}$, which cannot be simplified any farther. Note that this notation represents a single number—the distance around the perimeter of the triangle—even though it is written as the sum of two terms.

4. Explain how you could use this diagram to locate the two solutions to the quadratic equations given in the 2nd problem above: $2+\sqrt{2}$ and $2-\sqrt{2}$.

irrational



5. Are the numbers we have located on the number line in this way rational numbers or irrational numbers? Explain your answer.

fraction

π

cannot be expressed as a fraction.

Both sets of quadratic equations given in problems 1 and 2 above have solutions that can be plotted on a number line. The solutions to the first set of quadratic equations are rational numbers. The solutions to the 2nd set of quadratic equations are irrational numbers.

Big Idea #1: The set of numbers that contains all of the *rational numbers* and all of the *irrational numbers* is called the set of *real numbers*. The location of all points on a number line can be represented by real numbers.

Part 2: Imaginary and Complex Numbers

In the previous task, *To Be Determined . . .*, you found that the quadratic formula gives the solutions to the quadratic equation $x^2 + 4x + 5 = 0$ as $-2 + \sqrt{-1}$ and $-2 - \sqrt{-1}$. Because the square root of

$-2 + 1i$ $\quad$ $-2 - 1i$

a negative number has no defined value as either a rational or an irrational number, Euler proposed that a new number $i = \sqrt{-1}$ be including in what came to be known as the complex number system.

6. Based on Euler's definition of i, what would the value of i^2 be?

$$* \quad i^2 = \cancel{(\sqrt{-1})^2} = -1$$

$$i^3 = \cancel{\sqrt{-1} \cdot \sqrt{-1}} \cdot \sqrt{-1}$$

$$-1 \cdot \sqrt{-1} = -i$$

$$i^4 = i^2 \cdot i^2 = (-1)(-1) = 1$$

$$i^{20} = \underbrace{i^2 \cdot i^2}_{-1 \cdot -1} \cdot \underbrace{i^2 \cdot i^2}_{-1 \cdot -1} \cdot \underbrace{i^2 \cdot i^2}_{-1 \cdot -1} \cdot \underbrace{i^2 \cdot i^2}_{-1 \cdot -1} \cdot \underbrace{i^2 \cdot i^2}_{-1 \cdot -1}$$

$$1 \cdot 1 \quad 1 \quad 1 \quad 1$$

$$= \boxed{1}$$

With the introduction of the number i, the square root of *any* negative number can be represented. For example, $\sqrt{-2} = \sqrt{2}\cdot\sqrt{-1} = \sqrt{2}\cdot i$ and $\sqrt{-9} = \sqrt{9}\cdot\sqrt{-1} = 3i$.

7. Find the values of the following expressions. Show the details of your work.

(a) $\left(\sqrt{2}\cdot i\right)^2$ $(\sqrt{2})^2\cdot i^2$

$2(-1) = \boxed{-2}$

(b) $3i \times 3i$ $= 9i^2$

$9(-1) = \boxed{-9}$

Idea

$3x\cdot 3x$

$9x^2$

Using this new notation, the solutions to the equation $x^2+4x+5=0$ can be written as $-2+i$ and $-2-i$, and the factored form of x^2+4x+5 can be written as $(x+2-i)(x+2+i)$.

8. Verify that x^2+4x+5 and $(x+2-i)(x+2+i)$ are equivalent by expanding and simplifying the factored form. Show the details of your work.

	x	2	$-i$
x	x^2	$2x$	$-xi$
2	$2x$	4	$-2i$
i	xi	$2i$	1

$-i\cdot i$

$-i^2$

$-(-1)$

$= 1$

x^2+4x+5

Big Idea #2: Numbers like $3i$ and $\sqrt{2} \cdot i$ are called *pure imaginary numbers*. Numbers like $-2 - i$ and $-2 + i$ that include a real term and an imaginary term are called *complex numbers*.

The quadratic formula is usually written in the form $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. An equivalent form is $\dfrac{-b}{2a} \pm \dfrac{\sqrt{b^2 - 4ac}}{2a}$. If a, b and c are rational coefficients, then $\dfrac{-b}{2a}$ is a rational term, and $\dfrac{\sqrt{b^2 - 4ac}}{2a}$ may be a rational term, an irrational term or an imaginary term, depending on the value of the expression under the square root sign.

axis of symmetry

distance to x-intercepts

9. Examine the roots of the quadratic $y = x^2 - 6x + 7$ shown in the graph at the right. How do the terms $\frac{-b}{2a}$ and $\frac{\sqrt{b^2 - 4ac}}{2a}$ show up in this graph?

a b c

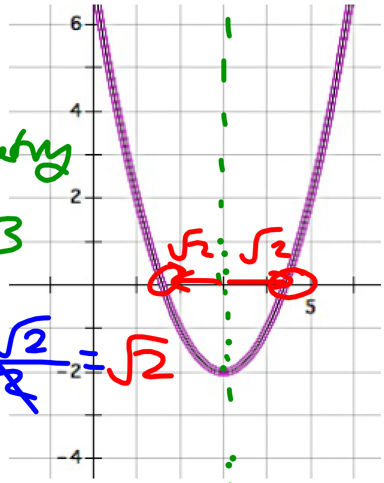
axis of symmetry

$\frac{--6}{2(1)} = \frac{6}{2} = 3$

distance

$\frac{\sqrt{(6)^2 - 4(1)(7)}}{2(1)} = \frac{\sqrt{36-28}}{2}$

$\frac{\sqrt{8}}{2} = \frac{\sqrt{4 \cdot 2}}{2} = \frac{\cancel{2}\sqrt{2}}{\cancel{2}} = \sqrt{2}$



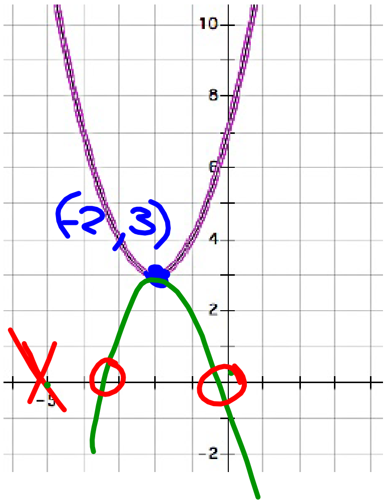
$x = 3$

Look back at the work you did in the task *To Be Determined . . .*

10. Which quadratics in that task had complex roots? (List them here.)

11. How can you determine if a quadratic has complex roots from its graph?

12. Find the complex roots of the following quadratic function represented by its graph.



$(x+2)^2 + \cancel{3} = 0$
$\quad\ \ \cancel{-3} \quad -3$

$\sqrt{(x+2)^2} = \sqrt{-3}$ ← discriminant determines if the result is real or imaginary

$x+2 = \pm\sqrt{3}\,i$
$\ \ -2 \quad -2$

$x = -2 \pm \sqrt{3}\,i$

13. Reflect the graph of the quadratic function given in question 12 over the horizontal line $y = 3$. Find the irrational roots of the reflected quadratic function.

$$-(x+2)^2 + \cancel{3} = \cancel{0}$$
$$\quad -3 \qquad -3$$
$$\cancel{-}(x+2)^2 = \cancel{-3}$$
$$-1 \qquad -1$$
$$\sqrt{(x+2)^2} = \pm\sqrt{3}$$

$$(x+2) = \pm\sqrt{3}$$
$$-2 \qquad -2$$
$$x = -2 \pm \sqrt{3}$$

14. How is the work you did to find the roots of the quadratic functions in questions 12 and 13 similar?

We solved the exact same way.
One resulted in a negative $\sqrt{}$
so it had imaginary solutions.

Big Idea #3: Complex numbers are not real numbers—they do not lie on the real number line that includes all of the rational and irrational numbers; also note that the real numbers are a subset of the complex numbers since a real number results when the imaginary part of $a + bi$ is 0, that is, $a + 0i$.

$2 + 0i$

<u>The Fundamental Theorem of Algebra, Revisited</u>

Remember the following information given in the previous task:

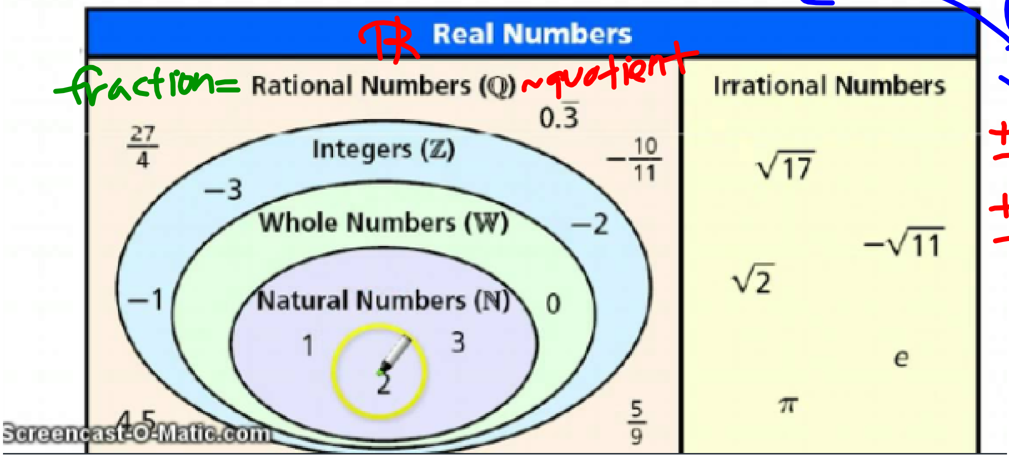
> A polynomial function is a function of the form:
>
> $$y = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots a_{n-3}x^3 + a_{n-2}x^2 + a_{n-1}x + a_n$$
>
> where all of the exponents are positive integers and all of the coefficients $a_0 \ldots a_n$ are constants.

As the theory of finding roots of polynomial functions evolved, a 17th century mathematician, Girard (1595-1632) made the following claim which has come to be known as <u>the Fundamental Theorem of Algebra</u>: *An nth degree polynomial function has n roots.*

15. Based on you work in this task, do you believe this theorem holds for quadratic functions? That is, do all functions of the form $y = ax^2 + bx + c$ always have two roots?

All numbers that can be represented on a number line are called **real numbers** and can be classified according to their characteristics.

Complex



$\pm i$ pure imaginary

$\pm 2i$

$2 - i$, $2 + i$

real + imaginary

$2 + 0i$

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3.9

READY, SET, GO! Name Period Date

READY

Topic: Classifying numbers according to set. Everything is complex!

Classify each of the numbers represented below according to the sets to which they belong. If a number fits in more than one set then list all that apply.

(Whole numbers "$\mathbb{W}$", Integers "$\mathbb{Z}$", Rational "$\mathbb{Q}$", Irrational "$\overline{\mathbb{Q}}$", Real "$\mathbb{R}$", Complex "$\mathbb{C}$")

1, 2, 3 -3, -1 $\frac{2}{3}$ $\sqrt{2}$ 2, $\sqrt{2}$, π

pure: $2i$
combo: $1+2i$

1. π $\mathbb{C}$ $\overline{\mathbb{Q}}$, $\mathbb{R}$

2. -13

3. $\sqrt{-16}$ $\mathbb{C}$ imaginary

4. 0

5. $\sqrt{75}$

6. $\frac{9}{3}$

7. $\sqrt{\frac{4}{9}}$ $\frac{\sqrt{4}}{\sqrt{9}} = \frac{2}{3}$ $\mathbb{Q}$, $\mathbb{R}$

8. $5+\sqrt{2}$ $\mathbb{C}$ $\mathbb{R}$ & $\overline{\mathbb{Q}}$ irrational
...forever
= irrational

9. $\sqrt{-40}$ pure imaginary $\mathbb{C}$

idea: $x + 2 + 2x + 3 = 3x + 5$

SET

Topic: Simplifying radicals, imaginary numbers

Simplify each radical expression below.

10. $3 + \sqrt{2} - 7 + 3\sqrt{2}$

$-4 + 4\sqrt{2}$

11. $\sqrt{5} - 9 + 8\sqrt{5} + 11 - \sqrt{5}$

12. $\sqrt{12} + \sqrt{48}$

13. $\sqrt{8} - \sqrt{18} + \sqrt{32}$

$2\sqrt{2} - 3\sqrt{2} + 4\sqrt{2}$

$\boxed{= 3\sqrt{2}}$

14. $11\sqrt{7} - 5\sqrt{7}$

15. $7\sqrt{7} + 5\sqrt{3} - 3\sqrt{7} + \sqrt{3}$

key

$i = \sqrt{-1}$

$i^2 = -1$

Simplify. Express as a complex number using "i" if necessary.

16. $\sqrt{-2} \cdot \sqrt{-2}$

$(\sqrt{2})^2 = -2$

17. $7 + \sqrt{-25}$ ~ $\sqrt{25}\,i$

$7 \pm 5i$

18. $(4i)^2$

19. $i^2 \cdot i^3 \cdot i^4 = i^9 = i \cdot i \cdot i \cdot i \cdot i$

$(i^2)^4 \cdot i^1$

$(-1)^4 \cdot i$

$+1 \cdot i = \boxed{i}$

20. $(\sqrt{-4})^3$

21. $(2i)(5i)(5i)$

$50i^3$

$50i^2 \cdot i$

$50(-1)i$

$\boxed{-50i}$

Solve each quadratic equation over the set of complex numbers.

$1x^2 + 0x + 100 = 0$

22. $x^2 + 100 = 0$

$-100 \quad -100$

$\sqrt{x^2} = \sqrt{-100}$

$\boxed{x = \pm 10i}$

~~100~~ CAN'T 0

23. $t^2 + 24 = 0$

24. $x^2 - 6x + 13 = 0$

25. $r^2 - 2r + 5 = 0$

GO determines real or not.

Topic: Solve Quadratic Equations $\rightarrow \sqrt{b^2 - 4ac}$ $\sqrt{+}$ real, $\sqrt{-}$ imaginary

Use the discriminant to determine the nature of the roots to the quadratic equation.

26. $x^2 - 5x + 7 = 0$ (a, b, c)

$\sqrt{(-5)^2 - 4(1)(7)}$

$25 - 28 = \sqrt{-3}$

$\mathbb{C}$ imaginary

27. $x^2 - 5x + 6 = 0$

$\sqrt{(-5)^2 - 4(1)(6)} = \sqrt{+1}$

$\mathbb{R}$

28. $2x^2 - 5x + 5 = 0$

29. $x^2 + 7x + 2 = 0$

30. $2x^2 + 7x + 6 = 0$

31. $2x^2 + 7x + 7 = 0$

32. $2x^2 - 7x + 6 = 0$

33. $2x^2 + 7x - 6 = 0$

34. $x^2 + 6x + 9 = 0$

Solve the quadratic equations below using an appropriate method.

35. $m^2 + 15m + 56 = 0$

36. $5x^2 - 3x + 7 = 0$

37. $x^2 - 10x + 21 = 0$

$(x-3)(x-7)$

$x = 3, 7$

38. $6x^2 + 7x - 5 = 0$

$6 \cdot -5 = -30$

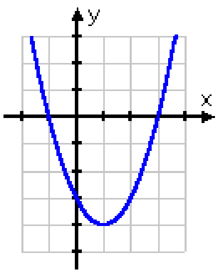
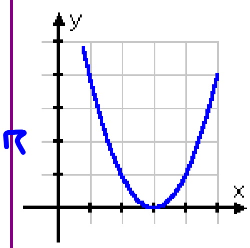
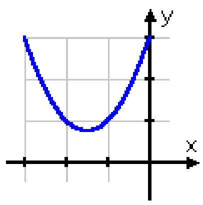
$0 = (3x+5)(2x-1)$

$3x + 5 = 0$, $3x = -5$, $x = -\frac{5}{3}$

$2x - 1 = 0$, $2x = 1$, $x = \frac{1}{2}$

	$3x$	$+5$
$2x$	$6x^2$	$10x$
-1	$-3x$	-5

Additional Examples

x^2-2x-3	x^2-6x+9	x^2+3x+3
$x=\frac{2\pm\sqrt{(-2)^2-4(-3)}}{2}$ $=\frac{2\pm\sqrt{4+12}}{2}$ $=\frac{2\pm\sqrt{16}}{2}=\frac{2\pm4}{2}$ $=\frac{-2}{2},\ \frac{6}{2}=-1,\ 3$	$x=\frac{6\pm\sqrt{(-6)^2-4(9)}}{2}$ $=\frac{6\pm\sqrt{36-36}}{2}$ $=\frac{6\pm\sqrt{0}}{2}=\frac{6\pm0}{2}=3$	$x=\frac{-3\pm\sqrt{(3)^2-4(3)}}{2}$ $=\frac{-3\pm\sqrt{9-12}}{2}$ $=\frac{-3\pm\sqrt{-3}}{2}$ $=-\frac{3}{2}\pm\frac{\sqrt{3}i}{2}$
a positive number inside the square root	zero inside the square root	a negative number inside the square root
two real solutions	one (repeated) real solution	two complex solutions
		
two distinct x-intercepts	one (repeated) x-intercept	no x-intercepts

i^2 is -1:

- **Simplify $2i + 3i$.**

  $2i + 3i = (2 + 3)i = \mathbf{5i}$

- **Simplify $16i - 5i$.**

  $16i - 5i = (16 - 5)i = \mathbf{11i}$

- **Multiply and simplify $(3i)(4i)$.**

  $(3i)(4i) = (3 \cdot 4)(i \cdot i) = (12)(i^2) = (12)(-1) = \mathbf{-12}$

- **Multiply and simplify $(i)(2i)(-3i)$.**

  $(i)(2i)(-3i) = (2 \cdot -3)(i \cdot i \cdot i) = (-6)(i^2 \cdot i)$

  $= (-6)(-1 \cdot i) = (-6)(-i) = \mathbf{6i}$

Note this last problem. Within it, you can see that $i^3 = -i$, because $i^2 = -1$. Continuing, we get:

$$i^4 = i^2 \cdot i^2 = (-1)(-1) = 1$$

This pattern of powers, signs, 1's, and i's is a cycle:

$$i^1 = i$$
$$i^2 = -1$$
$$i^3 = -i$$
$$i^4 = 1$$
$$i^5 = i^1 = i$$
$$i^6 = i^2 = -1$$
$$i^7 = i^3 = -i$$
$$i^8 = i^4 = 1$$

$$i^{99} = i^{96+3} = i^{(4\times24)+3} = i^3 = -i$$

That is, $i^{99} = i^3$, because you can just lop off the i^{96}. (Ninety-six is a multiple of four, so i^{96} is just 1, which you can ignore.) In other words, you can divide the exponent by 4 (using long division), discard the answer, and use only the remainder. This will give you the part of the exponent that you care about. Here are a few more examples:

- Simplify i^{17}.

  $i^{17} = i^{16+1} = i^{4\cdot4+1} = i^1 = i$

- Simplify i^{120}.

  $i^{120} = i^{4\cdot30} = i^{4\cdot30+0} = i^0 = 1$

- Simplify $i^{64,002}$.

  $i^{64,002} = i^{64,000+2} = i^{4\cdot16,000+2} = i^2 = -1$